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Multiplier Rules and Saddle-Point Theorems for Helbig's Approximate Solutions in Convex Pareto Problems

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Abstract. This paper deals with approximate Pareto solutions in convex multiobjective optimization problems. We relate two approximate Pareto efficiency concepts: one is already classic and the other is due to Helbig. We obtain Fritz John and Kuhn–Tucker type necessary and sufficient conditions for Helbig's approximate solutions. An application we deduce saddle-point theorems corresponding to these solutions for two vector-valued Lagrangian functions.

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1. Introduction

Fritz John and Kuhn–Tucker type rules are basic in optimization because they describe conditions for solutions in mathematical programs. Different authors have extended these rules to obtain conditions for approximate solutions in optimization problems.

In convex scalar optimization, it is possible to obtain multiplier rules for approximate solutions using the ε-subdifferential (Strodiot et al., 1983; Yokoyama, 1992). In nonconvex scalar optimization, the general method to obtain multiplier rules for approximate solutions is based on variational principles (Loridan, 1982).

Multiobjective optimization problems add an additional detail since in this kind of programs the notion of approximate efficient solution is not unique. The concept more used in the bibliography is due to Kutateladze

(1979) and Loridan (1984). Using this concept Liu (1991, 1996), Yokoyama (1996), Liu and Yokoyama (1999) and the authors (Gutiérrez et al., submitted) have obtained Kuhn–Tucker type conditions for approximate Pareto solutions. During these last years, several authors have already initiated the study of approximate solutions in optimization problems involving set-valued maps (see, for example, Rong and Wu, 2000).

In this work we provide multiplier rules for approximate solutions in convex multiobjective optimization problems using the ε-subdifferential and a notion of approximate efficiency due to Helbig (1992). With these rules we complete the Kuhn–Tucker type conditions above-mentioned.

As an application of these multiplier rules we develop saddle-point theorems corresponding to approximate solutions in the sense of Helbig using two vector-valued Lagrangian functions.

Our saddle-point theorems include approximate saddle-point results for scalar problems, extend by means of Helbig's approximate solutions similar results obtained for exact Pareto solutions in Tanino and Sawaragi (1979), Corley (1981), and Luc (1984) and give an answer to a problem formulated in Remark 3.3 of Vályi (1987).

Section 2 contains definitions and some results subsequently used. Moreover some connections between the concepts of Kutateladze–Loridan and Helbig are analyzed. Section 3 describes a general method to transform a multiobjective optimization problem in a scalar optimization problem in such a way that approximate Pareto solutions for the first problem are approximate solutions for the second problem. We next look into convex multiobjective optimization problems and we deduce multiplier rules for approximate Pareto solutions in the sense of Helbig. In Section 4 we deduce saddle-point theorems for approximate Pareto solutions in the sense of Helbig using the multiplier rules obtained in Section 3. Finally, Section 5 presents some conclusions that summarize this work.

2. Notation and Preliminaries

Let X be a normed space. In this paper we analyze the multiobjective optimization problem

$$\text{Min}\{f(x) | x \in K\}, \tag{1}$$

where $f: X \to \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_p)$ and $K \subset X$, $K \neq \emptyset$. We consider Pareto solutions for (1).

DEFINITION 2.1. A point $x_0 \in K$ is said to be an efficient Pareto solution (or Pareto solution) for (1), denoted $x_0 \in \text{Min}(f, K)$, if there is no $x \in K$ such that $f(x) - f(x_0) \in -\mathbb{R}^p_+ \backslash \{0\}$.

We write X^* for the topological dual of X. The interior of $M \subset X$ will be written by $\mathrm{int}(M)$. Let I_M denote the indicator function of M. For a linear map $A{:}X \to \mathbb{R}^m$ we use $\mathrm{Im}(A)$ and A^t to denote the image and the transpose map of A respectively. Finally, if $g{:}X \to \mathbb{R} \cup \{\infty\}$ is a convex function then $\mathrm{dom}(g)$ designates its domain.

Definition 2.2 establishes the notion of approximate solution for scalar programming.

DEFINITION 2.2. In problem (1), assume that $p = 1$. Let $\varepsilon \geqslant 0$. A point $x_0 \in K$ is said to be an ε-solution (or approximate solution up to ε) for (1), if $f(x_0) - \varepsilon \leqslant f(x)$, $\forall x \in K$.

The following definitions describe two concepts that extend the notion of approximate solution from scalar optimization problems to multiobjective programs.

DEFINITION 2.3 (Kutateladze, 1979; Loridan, 1984). Let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p) \in \mathbb{R}^p_+$. A point $x_0 \in K$ is said to be an $\bar{\varepsilon}$-efficient Pareto solution (or $\bar{\varepsilon}$-Pareto solution) for (1), denoted $x_0 \in \mathrm{Min}_{\bar{\varepsilon}}(f, K)$, if there is no $x \in K$ such that $f(x) - f(x_0) + \bar{\varepsilon} \in -\mathbb{R}^p_+ \backslash \{0\}$.

DEFINITION 2.4 (Helbig, 1992). Let $h \in \mathrm{int}(\mathbb{R}^p_+)$ and let $\varepsilon \geqslant 0$. A point $x_0 \in K$ is said to be an (ε, h)-efficient Pareto solution (or (ε, h)-Pareto solution) for (1), denoted $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, K)$, if

$$x \in K, f(x) - f(x_0) \in -\mathbb{R}^p_+ \Rightarrow \langle h, f(x_0)\rangle \leqslant \langle h, f(x)\rangle + \varepsilon.$$

Definition 2.4 was introduced by Helbig (1992) in the context of a linear topological space Y ordered by a nontrivial cone. Notice that Definitions 2.3 and 2.4 become Definition 2.1 when $\bar{\varepsilon} = 0$ and $\varepsilon = 0$, respectively. Moreover, when $p = 1$ Definition 2.3 and Definition 2.4 with $h = 1$ become Definition 2.2.

If problem (1) is Max then we define the sets of solutions $\mathrm{Max}(f, K)$, $\mathrm{Max}_{\bar{\varepsilon}}(f, K)$ and $\mathrm{Max}_{\varepsilon,h}(f, K)$ in a similar way as Definitions 2.1, 2.3 and 2.4, respectively.

In convex scalar optimization there exists a concept very useful to obtain information about approximate solutions: the ε-subdifferential.

DEFINITION 2.5 (Hiriart-Urruty, 1982). Let $g{:}X \to \mathbb{R} \cup \{\infty\}$ be a convex proper function. Let $x_0 \in \mathrm{dom}(g)$ and $\varepsilon \geqslant 0$. The ε-subdifferential of g at x_0 is the set $\partial_\varepsilon g(x_0)$ defined by

$$\partial_\varepsilon g(x_0) = \{x^* \in X^* | \langle x^*, x - x_0 \rangle \leqslant g(x) - g(x_0) + \varepsilon, \forall x \in X\}.$$

We use this concept in Section 3 to deduce multiplier rules for (ε, h)-Pareto solutions in convex multiobjective optimization problems.

The ε-subdifferential satisfies several calculus rules. Next we describe the formula for a sum of convex functions (see Theorem 2.1 in Hiriart-Urruty, 1982 for more details).

THEOREM 2.6. Let $g_1, g_2 : X \to \mathbb{R} \cup \{\infty\}$ *be two proper convex functions such that there exists* $x \in \mathrm{dom}(g_1)$ *at which* g_2 *is finite and continuous. Then* $\forall \varepsilon \geqslant 0$ *and* $\forall x_0 \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2)$,

$$\partial_\varepsilon (g_1 + g_2)(x_0) = \bigcup_{\substack{\varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial_{\varepsilon_1} g_1(x_0) + \partial_{\varepsilon_2} g_2(x_0)\}.$$

The following proposition describes the ε-subdifferential of an indicator function and motivates the notion of ε-normal set. For more details, we refer the reader to Hiriart-Urruty, 1982.

PROPOSITION 2.7. *Let* M *be a nonempty closed convex subset of* X *and let* $x_0 \in M$. *Then* $\partial_\varepsilon \mathrm{I}_M(x_0) = \{x^* \in X^* : \langle x^*, x - x_0 \rangle \leqslant \varepsilon, \ \forall x \in M\}$.

DEFINITION 2.8. Let M be a nonempty closed convex subset of X and let $x_0 \in M$. The set $\mathrm{N}_\varepsilon(M, x_0)$ of ε-normals to M at x_0 is defined by $\mathrm{N}_\varepsilon(M, x_0) = \partial_\varepsilon \mathrm{I}_M(x_0)$.

We finish this section giving some relationships between Definitions 2.3 and 2.4.

PROPOSITION 2.9. *Let* $\varepsilon \geqslant 0$, $\bar{\varepsilon} \in \mathbb{R}^p_+ \backslash \{0\}$ *and* $h \in \mathrm{int}(\mathbb{R}^p_+)$ *such that* $\langle h, \bar{\varepsilon} \rangle \geqslant 1$. *Approximate solutions for* (1) *satisfy the following relations*:

(i) $\mathrm{Min}(f, K) \subset \mathrm{Min}_{\varepsilon,h}(f, K) \subset \mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(f, K)$.
(ii) *If* $\varepsilon = 0$ *then* $\mathrm{Min}(f, K) = \mathrm{Min}_{\varepsilon,h}(f, K) = \mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(f, K)$.

Proof. (i) If $x_0 \in \mathrm{Min}(f, K)$ then $f(K) \cap (f(x_0) - \mathbb{R}^p_+) = \{f(x_0)\}$. Thus, for all $x \in K$ such that $f(x) - f(x_0) \in -\mathbb{R}^p_+$, $f(x) = f(x_0)$ and, consequently,

$$\langle h, f(x) \rangle + \varepsilon = \langle h, f(x_0) \rangle + \varepsilon \geqslant \langle h, f(x_0) \rangle.$$

Therefore $\mathrm{Min}(f, K) \subset \mathrm{Min}_{\varepsilon,h}(f, K)$.

Now, let $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, K)$ and suppose that $x_0 \notin \mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(f, K)$. Then there exists $x \in K$ such that

$$f(x) - (f(x_0) - \varepsilon \cdot \bar{\varepsilon}) \in -\mathbb{R}^p_+ \backslash \{0\}. \tag{2}$$

By (2), $f(x) - f(x_0) \in -\mathbb{R}^p_+$ and

$$\langle h, f(x_0)\rangle \leqslant \langle h, f(x)\rangle + \varepsilon \tag{3}$$

since $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, K)$. From (2) we have $f(x) + \varepsilon \cdot \bar{\varepsilon} - f(x_0) \in -\mathbb{R}^p_+ \backslash \{0\}$. As $h \in \mathrm{int}(\mathbb{R}^p_+)$ we obtain $\langle h, f(x) + \varepsilon \cdot \bar{\varepsilon} - f(x_0)\rangle < 0$. Since $\langle h, \bar{\varepsilon}\rangle \geqslant 1$ it follows that $\langle h, f(x)\rangle + \varepsilon < \langle h, f(x_0)\rangle$, contrary to (3).

(ii) A simple verification shows that $\mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(f, K) \subset \mathrm{Min}(f, K)$ if $\varepsilon = 0$. Then part (i) proves the inclusions in (ii). □

The following simple example shows that the last inclusion in Proposition 2.9(i) is not an equality in general.

EXAMPLE 2.10. Consider problem (1). Suppose that $X = \mathbb{R}^2$, $K = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leqslant 1\}$, $p = 2$ and $f(x, y) = (x, y)$. Let $\varepsilon = 1$, $\bar{\varepsilon} = (1, 0)$, $h = (1, 1)$ and $x_0 = (0, 0)$. It is clear that $x_0 \in \mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(f, K)$ since $f(x_0) - \varepsilon \cdot \bar{\varepsilon} = (-1, 0) \in \mathrm{Min}(f, K)$. However x_0 is not an (ε, h)-Pareto solution since $x = (-1/\sqrt{2}, -1/\sqrt{2}) \in K$, $f(x) - f(x_0) \in -\mathbb{R}^2_+$ and

$$\langle h, f(x)\rangle + \varepsilon = -\sqrt{2} + 1 < 0 = \langle h, f(x_0)\rangle.$$

3. Multiplier Rules for Approximate Pareto Solutions in Convex Multiobjective Programs

In this section we assume that the feasible set in problem (1) is $K = S \cap C$, with

$$S = \{x \in X | g_j(x) \leqslant 0, j = 1, 2, \ldots, m\},$$

$g_j: X \to \mathbb{R}$, $j = 1, 2, \ldots, m$ and $C \subset X$, $C \neq \emptyset$.

Let $\varepsilon \geqslant 0$, $h \in \mathrm{int}(\mathbb{R}^p_+)$ and $x_0 \in X$ be fixed. We can analyze if $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, K)$ by means of the scalar optimization problem

$$\mathrm{Min}\{F(x) | x \in C\}, \tag{4}$$

where $F(x) = \max\limits_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant m}} \{f_i(x) - f_i(x_0), g_j(x), \langle h, f(x)\rangle + \varepsilon - \langle h, f(x_0)\rangle\}$.

LEMMA 3.1. *Problems* (1) *and* (4) *satisfy the following relations.*

(i) *If* $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap C)$ *then* $x_0 \in \mathrm{Min}_{\varepsilon}(F, C)$.
(ii) *If* $0 \leqslant \delta < \varepsilon$ *and* $x_0 \in \mathrm{Min}_{\delta}(F, C) \cap S$ *then* $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap C)$.

Proof. (i) Suppose that $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap C)$. As $x_0 \in S$ and $\varepsilon \geqslant 0$ we have $F(x_0) = \varepsilon$. Moreover,

$$F(x) \geqslant 0, \quad \forall x \in C.$$

Indeed, if there exists $z \in C$ such that $F(z) < 0$ then $z \in S$, because

$$g_j(z) \leqslant F(z) < 0, \quad \forall j = 1, 2, \ldots, m,$$

and $f(z)$ is better than $f(x_0)$ in the sense of Pareto since

$$f_i(z) - f_i(x_0) \leqslant F(z) < 0, \quad \forall i = 1, 2, \ldots, p.$$

Consequently, $\langle h, f(x_0)\rangle \leqslant \langle h, f(z)\rangle + \varepsilon$ since $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap C)$. This contradicts the fact that $\langle h, f(z)\rangle + \varepsilon - \langle h, f(x_0)\rangle \leqslant F(z) < 0$.

In summary, $x_0 \in C$, $F(x_0) = \varepsilon$ and $F(x) \geqslant 0, \forall x \in C$; therefore $x_0 \in \mathrm{Min}_\varepsilon(F, C)$.

(ii) Let $\delta \in [0, \varepsilon)$ and suppose that $x_0 \in \mathrm{Min}_\delta(F, C) \cap S$. Then $F(x) \geqslant F(x_0) - \delta$, $\forall x \in C$. If we evaluate $F(x_0) - \delta$ and we use that $\delta < \varepsilon$, then we obtain

$$F(x_0) - \delta = \varepsilon - \delta > 0$$

and it follows that $\forall x \in C$,

$$\max_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant m}} \{f_i(x) - f_i(x_0), g_j(x), \langle h, f(x)\rangle + \varepsilon - \langle h, f(x_0)\rangle\} = F(x) \geqslant F(x_0) - \delta > 0. \tag{5}$$

If $x \in S$ then $g_j(x) \leqslant 0$, $j = 1, 2 \ldots, m$. Therefore, if we apply inequality (5) to any $x \in S \cap C$ we conclude that there exists $i \in \{1, 2, \ldots, p\}$ such that $f_i(x) > f_i(x_0)$ and $f(x) - f(x_0) \notin -\mathbb{R}^p_+$, or $f(x) - f(x_0) \in -\mathbb{R}^p_+$ and $\langle h, f(x)\rangle + \varepsilon > \langle h, f(x_0)\rangle$. Consequently, $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap C)$. □

Lemma 3.1 allows to obtain Fritz John type conditions for (ε, h)-Pareto solutions in general convex multiobjective optimization problems.

In the remainder of this paper we consider the multiobjective optimization problem

$$\mathrm{Min}\{f(x) | x \in S \cap Q \cap M\}, \tag{6}$$

where $Q = \{x \in X | Ax - b = 0\}$, $A : X \to \mathbb{R}^r$ is a continuous linear map, $b \in \mathbb{R}^r$ and $M \subset X$ is a nonempty closed convex set. Moreover we require f_i, $i = 1, 2, \ldots, p$, and g_j, $j = 1, 2, \ldots, m$, to be convex functions and we suppose that $S \cap Q \cap M \neq \emptyset$.

THEOREM 3.2. *Consider the optimization problem* (6). *Assume that* $Q \cap \mathrm{int}(M) \neq \emptyset$ *and* $x_0 \in S \cap Q \cap M$.

(i) *If* $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ *with* $h = (h_1, h_2, \ldots, h_p) \in \mathrm{int}(\mathbb{R}^p_+)$, *then there exist* $(\eta, v, \alpha) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ *and multipliers* $(\lambda, \mu, \gamma) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ *such that*

$$(\eta, \nu, \alpha, \lambda, \mu, \gamma) \geqslant 0, \tag{7}$$

$$\sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{m} \mu_j + \gamma = 1, \tag{8}$$

$$0 \in \sum_{i=1}^{p} \partial_{\eta_i}[(\lambda_i + \gamma h_i) f_i](x_0) + \sum_{j=1}^{m} \partial_{\nu_j}(\mu_j g_j)(x_0) + \mathrm{Im}(A^t) + N_\alpha(M, x_0), \tag{9}$$

$$\sum_{i=1}^{p} \eta_i + \sum_{j=1}^{m} \nu_j - \gamma\varepsilon + \alpha \leqslant \sum_{j=1}^{m} \mu_j g_j(x_0). \tag{10}$$

(ii) *Let* $\varepsilon > 0$ *and* $h \in \mathrm{int}(\mathbb{R}^p_+)$. *If there exist* $(\eta, \nu, \alpha, \lambda, \mu, \gamma) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ *satisfying conditions* (7)–(10) *with strict inequality in* (10), *then* $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$.

Previously we are going to prove a lemma.

LEMMA 3.3. *Consider the optimization problems* (4) *and* (6) *under the hypotheses of Theorem* 3.2 *and let* $\delta \geqslant 0$. *Then* $x_0 \in \mathrm{Min}_\delta(F, Q \cap M)$ *if and only if there exist* $(\eta, \nu, \alpha) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ and multipliers $(\lambda, \mu, \gamma) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ *such that* (7)–(9) *and*

$$\sum_{i=1}^{p} \eta_i + \sum_{j=1}^{m} \nu_j + (\varepsilon - \delta) - \gamma\varepsilon + \alpha \leqslant \sum_{j=1}^{m} \mu_j g_j(x_0) \tag{11}$$

are satisfied.

Proof. It is clear from Definition 2.5 that $x_0 \in \mathrm{Min}_\delta(F, Q \cap M)$ if and only if $0 \in \partial_\delta(F + I_Q + I_M)(x_0)$. As $Q \cap \mathrm{int}(M) \neq \emptyset$ we can apply Theorem 2.6 and we deduce that $0 \in \partial_\delta(F + I_Q + I_M)(x_0)$ if and only if there exist $\alpha_i \geqslant 0$, $i = 1, 2, 3$, such that $\alpha_1 + \alpha_2 + \alpha_3 = \delta$ and

$$0 \in \partial_{\alpha_1} F(x_0) + \partial_{\alpha_2} I_Q(x_0) + \partial_{\alpha_3} I_M(x_0). \tag{12}$$

In view of Definition 2.8, (12) becomes $0 \in \partial_{\alpha_1} F(x_0) + N_{\alpha_2}(Q, x_0) + N_{\alpha_3}(M, x_0)$. Therefore, $0 \in \partial_\delta(F + I_Q + I_M)(x_0)$ if and only if there exist $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$(\alpha_1, \alpha_2, \alpha_3) \geqslant 0, \tag{13a}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = \delta \tag{13b}$$

and

$$0 \in \partial_{\alpha_1} F(x_0) + N_{\alpha_2}(Q, x_0) + N_{\alpha_3}(M, x_0). \tag{13c}$$

According to Theorem 4.1 in Hiriart-Urruty (1982) we deduce that $x^* \in \partial_{\alpha_1} F(x_0)$ if and only if there exist $p+m+1$ nonnegative constants $\beta_1, \beta_2, \ldots, \beta_p$, $v_1, v_2, \ldots, v_m$, ξ and $p+m+1$ nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_p$, $\mu_1, \mu_2, \ldots, \mu_m$, γ with $\sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{m} \mu_j + \gamma = 1$, such that

$$x^* \in \sum_{i=1}^{p} \partial_{\beta_i}(\lambda_i f_i)(x_0) + \sum_{j=1}^{m} \partial_{v_j}(\mu_j g_j)(x_0) + \partial_{\xi}\left(\gamma \sum_{i=1}^{p} h_i f_i\right)(x_0) \tag{14}$$

and

$$\sum_{i=1}^{p} \beta_i + \sum_{j=1}^{m} v_j + \xi + \varepsilon - \sum_{j=1}^{m} \mu_j g_j(x_0) - \gamma\varepsilon = \alpha_1. \tag{15}$$

Theorem 2.6 implies

$$\partial_{\xi}\left(\gamma \sum_{i=1}^{p} h_i f_i\right)(x_0) = \bigcup_{\substack{\xi_i \geqslant 0, \\ \xi_1+\xi_2+\ldots+\xi_p=\xi}} \sum_{i=1}^{p} \partial_{\xi_i}(\gamma h_i f_i)(x_0). \tag{16}$$

Combining (14) and (15) with (16) we see that $x^* \in \partial_{\alpha_1} F(x_0)$ if and only if there exist $2p+m$ nonnegative constants $\beta_1, \beta_2, \ldots, \beta_p$, $\xi_1, \xi_2, \ldots, \xi_p$, $v_1, v_2, \ldots, v_m$ and $p+m+1$ nonnegative multipliers $\lambda_1, \lambda_2, \ldots, \lambda_p$, $\mu_1, \mu_2, \ldots, \mu_m$, γ with $\sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{m} \mu_j + \gamma = 1$, such that

$$x^* \in \sum_{i=1}^{p} \partial_{\beta_i}(\lambda_i f_i)(x_0) + \sum_{j=1}^{m} \partial_{v_j}(\mu_j g_j)(x_0) + \sum_{i=1}^{p} \partial_{\xi_i}(\gamma h_i f_i)(x_0) \tag{17}$$

and

$$\sum_{i=1}^{p} \beta_i + \sum_{j=1}^{m} v_j + \sum_{i=1}^{p} \xi_i + \varepsilon - \sum_{j=1}^{m} \mu_j g_j(x_0) - \gamma\varepsilon = \alpha_1. \tag{18}$$

If we write $\eta_i = \beta_i + \xi_i$, $i = 1, 2, \ldots, p$, then we conclude by Theorem 2.6 that (17) and (18) are equivalent, respectively, to

$$x^* \in \sum_{i=1}^{p} \partial_{\eta_i}[(\lambda_i + \gamma h_i) f_i](x_0) + \sum_{j=1}^{m} \partial_{v_j}(\mu_j g_j)(x_0) \tag{19}$$

and

$$\sum_{i=1}^{p} \eta_i + \sum_{j=1}^{m} v_j + \varepsilon - \sum_{j=1}^{m} \mu_j g_j(x_0) - \gamma\varepsilon = \alpha_1. \tag{20}$$

Finally (13), (19) and (20) imply that $0 \in \partial_\delta(F + I_Q + I_M)(x_0)$ if and only if there exist $\alpha_i \geqslant 0$, $i = 1, 2, 3$, with

$$\alpha_1 + \alpha_2 + \alpha_3 = \delta, \tag{21}$$

$p + m$ nonnegative constants $\eta_1, \eta_2, \ldots, \eta_p$, $v_1, v_2, \ldots, v_m$ and $p + m + 1$ nonnegative multipliers, $\lambda_1, \lambda_2, \ldots, \lambda_p, \mu_1, \mu_2, \ldots, \mu_m, \gamma$ with $\sum_{i=1}^p \lambda_i + \sum_{j=1}^m \mu_j + \gamma = 1$, such that

$$0 \in \sum_{i=1}^p \partial_{\eta_i}[(\lambda_i + \gamma h_i)f_i](x_0) + \sum_{j=1}^m \partial_{v_j}(\mu_j g_j)(x_0) + \mathrm{N}_{\alpha_2}(Q, x_0) + \mathrm{N}_{\alpha_3}(M, x_0)$$

and

$$\sum_{i=1}^p \eta_i + \sum_{j=1}^m v_j + \varepsilon - \sum_{j=1}^m \mu_j g_j(x_0) - \gamma\varepsilon = \alpha_1. \tag{22}$$

Substituting (22) into (21) and using that $\alpha_2 \geqslant 0$ we obtain (11) with $\alpha = \alpha_3$. Now, it is well-known that $\mathrm{N}_{\alpha_2}(Q, x_0) = \mathrm{N}_0(Q, x_0) = (\mathrm{Ker}A)^\perp = \mathrm{Im}(A^t)$ (see Sections 5.7 and 6.6 in Luenberger, 1969 for more details) and the necessary condition is complete.

Conversely, assume that (7)–(9) and (11) are satisfied. We define α_1 by equality (22), $\alpha_2 = \delta - (\alpha_1 + \alpha)$ and $\alpha_3 = \alpha$. The constants α_1 and α_2 are nonnegative. Indeed, $\varepsilon - \gamma\varepsilon \geqslant 0$ since $\gamma \leqslant 1$ by (8). From $x_0 \in S$ it follows that $-\sum_{j=1}^m \mu_j g_j(x_0) \geqslant 0$, so $\alpha_1 \geqslant 0$. Finally, (11) shows that $\alpha_1 + \alpha \leqslant \delta$, and the sufficient condition is proved. □

Proof of Theorem 3.2. (i) By statement (i) of Lemma 3.1, if $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ then $x_0 \in \mathrm{Min}_\varepsilon(F, Q \cap M)$. The result of part (i) follows from Lemma 3.3 with $\delta = \varepsilon$. Notice that (11) becomes (10) when $\delta = \varepsilon$.

(ii) Reasoning as the last part of the proof of Lemma 3.3, we have that $\alpha_1 \geqslant 0$, where α_1 is defined by (22). So, $\alpha_1 + \alpha \geqslant 0$. Using that (10) is satisfied with strict inequality, it follows that $\alpha_1 + \alpha < \varepsilon$. Choosing δ such that $\alpha_1 + \alpha \leqslant \delta < \varepsilon$, then (11) is satisfied. The conclusion follows from Lemma 3.3 and statement (ii) of Lemma 3.1. □

REMARK 3.4. (i) (10) is a complementary slackness condition. Notice that if $\varepsilon = 0$ then $\mu_j g_j(x_0) = 0$, $j = 1, 2, \ldots, m$.

(ii) By Proposition 2.9, the multiplier rules in Theorem 3.2(i) are Fritz John type necessary conditions for Pareto solutions in problem (6). In the same way, the multiplier rules in Theorem 3.2(ii) are Fritz John type sufficient conditions for $\varepsilon \cdot \bar{\varepsilon}$-Pareto solutions in problem (6) if $\bar{\varepsilon} \in \mathbb{R}^p_+ \backslash \{0\}$ and $\langle h, \bar{\varepsilon} \rangle \geqslant 1$.

If problem (6) satisfies the so-called Slater constraint qualification:

(SCQ) $\exists z \in M$ such that $g(z) < 0$ and $Az = b$,

then multipliers $\lambda_i, i = 1, 2, \ldots, p$ and γ cannot all be zero and we obtain Kuhn–Tucker type conditions. The proof for Proposition 3.5 is similar to that of Proposition 4.1 in Gutiérrez et al. (submitted) and is omitted. This proposition and Theorem 3.2 extend Theorem 2.4 in Strodiot et al. (1983) taking $\varepsilon \cdot h$ as error.

PROPOSITION 3.5. *Consider problem* (6). *Let* $x_0 \in S \cap Q \cap M$. *Under the constraint qualification* (SCQ), *if* x_0 *satisfies conditions* (7)–(10) *then* $(\lambda, \gamma) \neq 0$.

Finally we use the multiplier rules attained in Theorem 3.2(ii) to obtain (ε, h)-Pareto solutions in one particular case.

EXAMPLE 3.6. With the notation of problem (6) suppose that $X = \mathbb{R}^2$, $p = 2$, $f(x, y) = (x, y)$, $m = 2$, $g_1(x, y) = -3x - y$, $g_2(x, y) = -x - 3y$ and $Q = M = \mathbb{R}^2$. Let $\varepsilon = 1$ and $h = (1, 1)$. It is clear that $\forall (x, y) \in \mathbb{R}^2$, $\forall \lambda_1, \lambda_2, \gamma, \mu_1, \mu_2 \in \mathbb{R}$ and $\forall \eta_1, \eta_2, v_1, v_2 \geqslant 0$,

$$\partial_{\eta_1}[(\lambda_1 + \gamma)f_1](x, y) = \{(\lambda_1 + \gamma, 0)\}, \quad \partial_{\eta_2}[(\lambda_2 + \gamma)f_2](x, y) = \{(0, \lambda_2 + \gamma)\},$$

$$\partial_{v_1}(\mu_1 g_1)(x, y) = \{\mu_1(-3, -1)\}, \quad \partial_{v_2}(\mu_2 g_2)(x, y) = \{\mu_2(-1, -3)\}.$$

Then Theorem 3.2(ii) implies that the following conditions give (1,(1,1))-Pareto solutions:

$$(\eta_1, \eta_2, v_1, v_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma) \geqslant 0, \tag{23}$$

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \gamma = 1, \tag{24}$$

$$(\lambda_1 + \gamma, \lambda_2 + \gamma) = \mu_1(3, 1) + \mu_2(1, 3), \tag{25}$$

$$\eta_1 + \eta_2 + v_1 + v_2 - \gamma < \mu_1(-3x - y) + \mu_2(-x - 3y), \tag{26}$$

$$-3x - y \leqslant 0, \quad -x - 3y \leqslant 0. \tag{27}$$

If we replace (26) by

$$\eta_1 + \eta_2 + v_1 + v_2 - \gamma \leqslant \mu_1(-3x - y) + \mu_2(-x - 3y), \tag{28}$$

then Theorem 3.2(i) shows that the above conditions are necessary for (1,(1,1))-Pareto solutions. We can certainly assume that $\eta_1 = \eta_2 = v_1 = v_2 = 0$ because if $\eta_1, \eta_2, v_1, v_2 \geqslant 0$ we have multipliers $\lambda_1, \lambda_2, \mu_1, \mu_2, \gamma$ and a point (x, y) such that $(\eta_1, \eta_2, v_1, v_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, x, y)$ satisfy (23)–(27) then $(0, 0, 0, 0, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma, x, y)$ solve (23)–(27) too.

Equations (24) and (25) imply $\lambda_1 = 1 - 2\mu_1 - 4\mu_2$, $\lambda_2 = 1 - 4\mu_1 - 2\mu_2$ and $\gamma = 5\mu_1 + 5\mu_2 - 1$. As λ_1, λ_2 and γ are nonnegative we have the following constraints for the multipliers μ_1 and μ_2:

$$\mu_1 \geqslant 0, \quad \mu_2 \geqslant 0, \quad 2\mu_1 + 4\mu_2 \leqslant 1, \quad 4\mu_1 + 2\mu_2 \leqslant 1, \quad 5\mu_1 + 5\mu_2 \geqslant 1. \tag{29}$$

Choosing $\mu = (1/4, 0)$, $\mu = (0, 1/4)$ and $\mu = (1/6, 1/6)$ respectively (three extremal points of the set defined by (29)) we obtain from (26) and (27) that

$$S \cap \{(x, y) \in \mathbb{R}^2 | 3x + y < 1 \quad \text{or} \quad x + 3y < 1 \quad \text{or} \quad x + y < 1\}$$

is a set of (1,(1,1))-Pareto solutions. Taking into account condition (28) we deduce that points in

$$S \cap \{(x, y) \in \mathbb{R}^2 | 3x + y \leqslant 1 \quad \text{or} \quad x + 3y \leqslant 1 \quad \text{or} \quad x + y \leqslant 1\}$$

satisfy necessary conditions for (1,(1,1))-Pareto solutions. It is easy to check that this is the set of (1,(1,1))-Pareto solutions.

4. Approximate Saddle-Point Theorems

As an application of Theorem 3.2 we deduce approximate saddle-point theorems for Helbig's approximate solutions in convex Pareto problems.

Let $k \geqslant 1$ be an integer and let $d \in \mathbb{R}^p_+$ be a fixed vector. We denote by L_k the linear space of linear functions from $\mathbb{R}^k$ into $\mathbb{R}^p$ and by $L_{k,d}$ the linear space of functions $\{\tilde{s} : \mathbb{R}^k \to \mathbb{R}^p | \tilde{s}(z) = \langle s, z \rangle d\}$ (notice that $s \in \mathbb{R}^k$ defines $\tilde{s}$). Let $L_k^+ = \{\tilde{s} \in L_k | \tilde{s}(\mathbb{R}^k_+) \subset \mathbb{R}^p_+\}$ and $L_{k,d}^+ = \{\tilde{s} \in L_{k,d} | s \in \mathbb{R}^k_+\}$. Both L_k^+ and $L_{k,d}^+$ are convex cones. Moreover it is clear that $L_{k,d} \subset L_k$ and $L_{k,d}^+ \subset L_k^+$.

According to these spaces we consider two vector-valued Lagrangian functions for (6): $\Phi_{d_1,d_2} : X \times L_{m,d_1} \times L_{r,d_2} \to \mathbb{R}^p \cup \{\pm\infty\}$ defined by the equality

$$\Phi_{d_1,d_2}(x, \tilde{s}^1, \tilde{s}^2) = \begin{cases} \infty & \text{if } x \notin M \\ f(x) + \tilde{s}^1(g(x)) + \tilde{s}^2(A(x) - b) & \text{if } x \in M, \tilde{s}^1 \in L^+_{m,d_1} \\ -\infty & \text{if } x \in M, \tilde{s}^1 \notin L^+_{m,d_1} \end{cases}$$

and $\Psi : X \times L_m \times L_r \to \mathbb{R}^p \cup \{\pm\infty\}$ defined similarly.

DEFINITION 4.1. Let $\bar{\varepsilon} \in \mathbb{R}^p_+$. A point $(x_0, \tilde{s}_0^1, \tilde{s}_0^2) \in X \times L_{m,d_1} \times L_{r,d_2}$ is said to be an $\bar{\varepsilon}$-Pareto saddle-point for the Lagrangian function Φ_{d_1,d_2} if:

(i) $x_0 \in \text{Min}_{\bar{\varepsilon}}(\Phi_{d_1,d_2}(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$;
(ii) $(\tilde{s}_0^1, \tilde{s}_0^2) \in \text{Max}_{\bar{\varepsilon}}(\Phi_{d_1,d_2}(x_0, \cdot, \cdot), L_{m,d_1} \times L_{r,d_2})$.

In Definition 4.1 we supplement the space $\mathbb{R}^p$ with the elements ∞ and $-\infty$ and we assume that the usual algebraic and ordering properties hold. The definition of $\bar{\varepsilon}$-Pareto saddle-point for the Lagrangian Ψ is similar.

Proposition 4.2 characterizes $\bar{\varepsilon}$-Pareto saddle-points for the Lagrangians Φ_{d_1,d_2} and Ψ. The proof is analogous to the demonstration of Proposition 3.1 in Vályi (1987) and it is omitted.

PROPOSITION 4.2.

(i) *Let* $\bar{\varepsilon} \in \mathbb{R}^p_+$ *and* $d_1, d_2 \in \bar{\varepsilon} + \mathbb{R}^p_+ \backslash \{0\}$. *A point* $(x_0, \tilde{s}_0^1, \tilde{s}_0^2) \in X \times L_{m,d_1} \times L_{r,d_2}$ *is an* $\bar{\varepsilon}$*-Pareto saddle-point for the Lagrangian* Φ_{d_1,d_2} *if and only if*:
  (a) $x_0 \in \mathrm{Min}_{\bar{\varepsilon}}(\Phi_{d_1,d_2}(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$,
  (b) $x_0 \in S \cap Q \cap M$,
  (c) $\tilde{s}_0^1(g(x_0)) + \bar{\varepsilon} \notin -\mathbb{R}^p_+ \backslash \{0\}$.

(ii) *The same is true for the Lagrangian* Ψ *in place of* Φ_{d_1,d_2} *if* $(x_0, \tilde{s}_0^1, \tilde{s}_0^2) \in X \times L_m \times L_r$ *and we replace* (a) *by*
  (a′) $x_0 \in \mathrm{Min}_{\bar{\varepsilon}}(\Psi(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$.

Next we show that in convex Pareto problems satisfying (SCQ), if $\varepsilon > 0$ and $\bar{\varepsilon}, h \in \mathrm{int}(\mathbb{R}^p_+)$ with $\langle h, \bar{\varepsilon} \rangle \geqslant 1$ then for every (ε, h)-Pareto solution x_0 there is a Lagrangian function Φ_{d_1,d_2} such that x_0 is an $\varepsilon \cdot \bar{\varepsilon}$-Pareto saddle-point of this function. We previously state an approximate version of the Lagrangian multiplier theorem under appropriate regularity conditions. This result extends Theorem 4.1 in Tanino and Sawaragi (1979), Theorem 2 in Corley (1981) and Theorem 3.2 in Luc (1984).

THEOREM 4.3. *Let* $\varepsilon, c > 0$ and $\bar{\varepsilon}$, h, $d_2 \in \mathrm{int}(\mathbb{R}^p_+)$ *such that* $\langle h, \bar{\varepsilon} \rangle \geqslant 1$. *Suppose that (SCQ) holds. If* $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ *then there exist* $\tilde{s}_0^1 \in L_{m,c\varepsilon\bar{\varepsilon}}$ *and* $\tilde{s}_0^2 \in L_{r,d_2}$ *such that* $x_0 \in \mathrm{Min}_{\varepsilon \cdot \bar{\varepsilon}}(\Phi_{c\varepsilon\bar{\varepsilon},d_2}(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$ *and* $\tilde{s}_0^1(g(x_0)) + \varepsilon \cdot \bar{\varepsilon} \in \mathbb{R}^p_+$.

Proof. As $x_0 \in \mathrm{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ it follows by Theorem 3.2 that there exists $(\eta, v, \alpha, \lambda, \mu, \gamma) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}$ such that (7)–(10) hold.

From (9) we deduce that there exist $x_i^* \in \partial_{\eta_i}[(\lambda_i + \gamma h_i) f_i](x_0)$, $i = 1, 2, \ldots, p$, $z_j^* \in \partial_{v_j}(\mu_j g_j)(x_0)$, $j = 1, 2, \ldots, m$, $a^* \in \mathrm{Im}(A^t)$ and $w^* \in N_\alpha(M, x_0)$ such that

$$\sum_{i=1}^{p} x_i^* + \sum_{j=1}^{m} z_j^* + a^* + w^* = 0. \tag{30}$$

As $a^* \in \mathrm{Im}(A^t)$ it follows that there exists $v = (v_1, v_2, \ldots, v_r)$ such that $a^* = A^t v$. Let us show that $x_0 \in \mathrm{Min}_{\gamma\varepsilon}(\langle \lambda + \gamma h, f(\cdot) \rangle + \langle \mu, g(\cdot) \rangle + \langle v, A(\cdot) - b \rangle, M)$. Indeed, by the definition of ε-subdifferential and ε-normal we obtain

$$(\lambda_i+\gamma h_i)f_i(x) \geqslant (\lambda_i+\gamma h_i)f_i(x_0) - \eta_i + \langle x_i^*, x - x_0\rangle, \quad i=1,2,\ldots,p, \quad \forall x \in X, \tag{31}$$

$$\mu_j g_j(x) \geqslant \mu_j g_j(x_0) - v_j + \langle z_j^*, x - x_0\rangle, \quad j = 1,2,\ldots,m, \quad \forall x \in X, \tag{32}$$

$$\alpha \geqslant \langle w^*, x - x_0\rangle, \quad \forall x \in M, \tag{33}$$

$$\langle v, A(x) - b\rangle = \langle v, A(x_0) - b\rangle + \langle a^*, x - x_0\rangle, \quad \forall x \in X. \tag{34}$$

Adding the inequalities in (31)–(34) and applying (30) we obtain $\forall x \in M$,

$$\langle \lambda + \gamma h, f(x)\rangle + \langle \mu, g(x)\rangle + \langle v, A(x) - b\rangle \geqslant \langle \lambda + \gamma h, f(x_0)\rangle + \langle \mu, g(x_0)\rangle + \langle v, A(x_0) - b\rangle - \sum_{i=1}^{p} \eta_i - \sum_{j=1}^{m} v_j - \alpha. \tag{35}$$

Applying (10) into (37) it follows that $\forall x \in M$,

$$\begin{aligned}&\langle \lambda + \gamma h, f(x)\rangle + \langle \mu, g(x)\rangle + \langle v, A(x) - b\rangle \geqslant \langle \lambda + \gamma h, f(x_0)\rangle + \\ &\langle \mu, g(x_0)\rangle + \langle v, A(x_0) - b\rangle - \sum_{j=1}^{m} \mu_j g_j(x_0) - \gamma\varepsilon \geqslant \\ &\langle \lambda + \gamma h, f(x_0)\rangle + \langle \mu, g(x_0)\rangle + \langle v, A(x_0) - b\rangle - \gamma\varepsilon,\end{aligned} \tag{36}$$

where the last inequality holds since $x_0 \in S$ and $\mu_j \geqslant 0$, $\forall j = 1,2,\ldots,m$. By (36) we see that

$$x_0 \in \mathrm{Min}_{\gamma\varepsilon}(\langle \lambda + \gamma h,\ f(\cdot)\rangle + \langle \mu, g(\cdot)\rangle + \langle v, A(\cdot) - b\rangle, M). \tag{37}$$

Let us consider $\tilde{s}_0^1 = \langle \beta_1 \mu, \cdot\rangle c\varepsilon\bar{\varepsilon} \in L_{m,c\varepsilon\bar{\varepsilon}}$ and $\tilde{s}_0^2 = \langle \beta_2 v, \cdot\rangle d_2 \in L_{r,d_2}$ with

$$\beta_1 = \frac{1}{\langle \lambda + \gamma h, c\varepsilon\bar{\varepsilon}\rangle}, \quad \beta_2 = \frac{1}{\langle \lambda + \gamma h, d_2\rangle}. \tag{38}$$

According to Proposition 3.5 we have $(\lambda, \gamma) \neq 0$. Then $\beta_1, \beta_2 > 0$ since $c, \varepsilon > 0$ and $h, \bar{\varepsilon},\ d_2 \in \mathrm{int}(\mathbb{R}_+^p)$. The proof is completed if we show that $x_0 \in \mathrm{Min}_{\varepsilon\cdot\bar{\varepsilon}}(\Phi_{c\varepsilon\bar{\varepsilon},d_2}(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$ and $\tilde{s}_0^1(g(x_0)) + \varepsilon \cdot \bar{\varepsilon} \in \mathbb{R}_+^p$.

Suppose, contrary to our claim, that $x_0 \notin \mathrm{Min}_{\varepsilon\cdot\bar{\varepsilon}}(\Phi_{c\varepsilon\bar{\varepsilon},d_2}(\cdot, \tilde{s}_0^1, \tilde{s}_0^2), X)$. Then we could find a point $x \in X$ such that

$$\Phi_{c\varepsilon\bar{\varepsilon},d_2}(x, \tilde{s}_0^1, \tilde{s}_0^2) - (\Phi_{c\varepsilon\bar{\varepsilon},d_2}(x_0, \tilde{s}_0^1, \tilde{s}_0^2) - \varepsilon \cdot \bar{\varepsilon}) \in -\mathbb{R}_+^p \backslash \{0\}. \tag{39}$$

$x_0 \in M$ and $\tilde{s}_0^1 \in L_{m,c\varepsilon\bar{\varepsilon}}^+$ since $\beta_1 > 0$ and $\mu \in \mathbb{R}_+^m$. Then $x \in M$ and we can rewrite (39) as

$$f(x) + \tilde{s}_0^1(g(x)) + \tilde{s}_0^2(A(x) - b) - (f(x_0) + \tilde{s}_0^1(g(x_0)) + \tilde{s}_0^2(A(x_0) - b) - \varepsilon \cdot \bar{\varepsilon}) \in -\mathbb{R}_+^p \backslash \{0\}.$$

Since $\lambda + \gamma h \in \mathbb{R}^p_+$ we have

$$\begin{aligned} &\langle \lambda + \gamma h, f(x) + \tilde{s}_0^1(g(x)) + \tilde{s}_0^2(A(x) - b) - (f(x_0) + \tilde{s}_0^1(g(x_0)) \\ &+ \tilde{s}_0^2(A(x_0) - b) - \varepsilon \cdot \bar{\varepsilon})\rangle \leqslant 0. \end{aligned} \tag{40}$$

By (38) we have $\langle \lambda + \gamma h, \tilde{s}_0^1(\cdot)\rangle = \langle \mu, \cdot\rangle$ and $\langle \lambda + \gamma h, \tilde{s}_0^2(\cdot)\rangle = \langle v, \cdot\rangle$. Using that $\langle h, \bar{\varepsilon}\rangle \geqslant 1$ we conclude from (40) that

$$\begin{aligned} &\langle \lambda + \gamma h, f(x)\rangle + \langle \mu, g(x)\rangle + \langle v, A(x) - b\rangle \leqslant \langle \lambda + \gamma h, f(x_0)\rangle + \langle \mu, g(x_0)\rangle \\ &+ \langle v, A(x_0) - b\rangle - \gamma\varepsilon - \langle \lambda, \varepsilon \cdot \bar{\varepsilon}\rangle. \end{aligned} \tag{41}$$

Inequalities (40) and (41) are strict if $\gamma \neq 0$ because $h \in \operatorname{int}(\mathbb{R}^p_+)$. Moreover $\langle \lambda, \varepsilon \cdot \bar{\varepsilon}\rangle > 0$ if $\lambda \neq 0$ since $\varepsilon \cdot \bar{\varepsilon} \in \operatorname{int}(\mathbb{R}^p_+)$. Then for $x \in M$ we have

$$\begin{aligned} &\langle \lambda + \gamma h, f(x)\rangle + \langle \mu, g(x)\rangle + \langle v, A(x) - b\rangle < \langle \lambda + \gamma h, f(x_0)\rangle + \langle \mu, g(x_0)\rangle \\ &+ \langle v, A(x_0) - b\rangle - \gamma\varepsilon \end{aligned}$$

in contradiction to (36).

Finally, we show that $\tilde{s}_0^1(g(x_0)) + \varepsilon \cdot \bar{\varepsilon} \in \mathbb{R}^p_+$. Indeed, by (38) we have

$$\tilde{s}_0^1(g(x_0)) + \varepsilon \cdot \bar{\varepsilon} = \langle \beta_1 \mu, g(x_0)\rangle c\varepsilon\bar{\varepsilon} + \varepsilon \cdot \bar{\varepsilon} = \left(\frac{\langle \mu, g(x_0)\rangle}{\langle \lambda + \gamma h, \varepsilon \cdot \bar{\varepsilon}\rangle} + 1\right)\varepsilon \cdot \bar{\varepsilon}$$

and $\tilde{s}_0^1(g(x_0)) + \varepsilon \cdot \bar{\varepsilon} \in \mathbb{R}^p_+$ if and only if $\langle \mu, g(x_0)\rangle + \langle \lambda + \gamma h, \varepsilon \cdot \bar{\varepsilon}\rangle \geqslant 0$. As $\langle h, \bar{\varepsilon}\rangle \geqslant 1$ and $\langle \lambda, \varepsilon \cdot \bar{\varepsilon}\rangle \geqslant 0$ we conclude from (10)

$$\langle \mu, g(x_0)\rangle + \langle \lambda + \gamma h, \varepsilon \cdot \bar{\varepsilon}\rangle \geqslant \sum_{j=1}^m \mu_j g_j(x_0) + \gamma\varepsilon \geqslant \sum_{i=1}^p \eta_i + \sum_{j=1}^m v_j + \alpha \geqslant 0.$$

□

Next we show that the conditions described in Theorem 4.3 are not sufficient for (ε, h)-Pareto efficiency.

EXAMPLE 4.4. In Example 2.10 the feasible point $(0, 0)$ is not an (1,(1,1))-Pareto solution. However, let $c = 1$, $\bar{\varepsilon} = (1, 1)$ and $\tilde{s}_0^1(z) = (z, z), \forall z \in \mathbb{R}$. It is clear that $\tilde{s}_0^1 \in L_{1,(1,1)}$ and $\tilde{s}_0^1(g_1(0, 0)) + \varepsilon \cdot \bar{\varepsilon} = (0, 0) \in \mathbb{R}^2_+$. Moreover $(0, 0) \in \operatorname{Min}_{(1,1)}(f(\cdot) + \tilde{s}_0^1(g_1(\cdot)), \mathbb{R}^2)$. In fact, suppose that there exists $(x, y) \in \mathbb{R}^2$ such that $f(x, y) + \tilde{s}_0^1(g_1(x, y)) \quad -(f(0, 0) + \tilde{s}_0^1(g_1(0, 0)) -(1, 1)) \in -\mathbb{R}^2_+\backslash\{0\}$. Then $(x + x^2 + y^2 + 1, y + x^2 + y^2 + 1) \in -\mathbb{R}^2_+\backslash\{0\}$ and this is a contradiction since $(x + x^2 + y^2 + 1, y + x^2 + y^2 + 1) = ((x + 1/2)^2 + y^2 + 3/4, x^2 + (y + 1/2)^2 + 3/4) \in \mathbb{R}^2_+$.

The following corollary is immediate from Proposition 4.2(i) and Theorem 4.3.

COROLLARY 4.5. *Consider problem* (6) *and suppose that* (*SCQ*) *holds. Let* $\varepsilon > 0$, $c > 1$, $\bar{\varepsilon}, h \in \text{int}(\mathbb{R}^p_+)$ *with* $\langle h, \bar{\varepsilon}\rangle \geqslant 1$ and $d_2 \in \varepsilon\bar{\varepsilon} + \mathbb{R}^p_+ \backslash \{0\}$. *If* $x_0 \in \text{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ *then there exist* $\tilde{s}^1_0 \in L_{m,c\varepsilon\bar{\varepsilon}}$ *and* $\tilde{s}^2_0 \in L_{r,d_2}$ *such that* $(x_0, \tilde{s}^1_0, \tilde{s}^2_0)$ *is an* $\varepsilon \cdot \bar{\varepsilon}$*-Pareto saddle-point for the vector-valued Lagrangian function* $\Phi_{c\varepsilon\bar{\varepsilon},d_2}$.

REMARK 4.6. We have chosen the Lagrangian Φ_{d_1,d_2} instead of Ψ to obtain necessary conditions for (ε, h)-Pareto solutions through its $\bar{\varepsilon}$-Pareto saddle-points because the space $L_{k,d}$ has less variables than the space L_k. This feature is important from the point of view of solving practical problems.

Nevertheless the corresponding result for the Lagrangian Ψ also holds and it is stated in the next corollary. Its proof is clear from Theorem 4.3 and Proposition 4.2(ii) since $L_{k,d} \subset L_k$.

COROLLARY 4.7. *Consider problem* (6) *and suppose that* (*SCQ*) *holds. Let* $\varepsilon > 0$ *and* $\bar{\varepsilon}, h \in \text{int}(\mathbb{R}^p_+)$ *with* $\langle h, \bar{\varepsilon}\rangle \geqslant 1$. *If* $x_0 \in \text{Min}_{\varepsilon,h}(f, S \cap Q \cap M)$ *then there exist* $\tilde{s}^1_0 \in L_m$ and $\tilde{s}^2_0 \in L_r$ *such that* $(x_0, \tilde{s}^1_0, \tilde{s}^2_0)$ *is an* $\varepsilon \cdot \bar{\varepsilon}$*-Pareto saddle-point for the vector-valued Lagrangian function* Ψ.

For the same reason in Remark 4.6 we should choose Ψ instead of Φ_{d_1,d_2} to obtain sufficient conditions for approximate Pareto solutions through its $\bar{\varepsilon}$-Pareto saddle-points.

PROPOSITION 4.8. *Let* $h \in \text{int}(\mathbb{R}^p_+)$, $\bar{\varepsilon} \in \mathbb{R}^p_+$ *and* $(x_0, \tilde{s}^1_0, \tilde{s}^2_0) \in X \times L^+_m \times L_r$. *Assume that* x_0 *is an approximate solution up to* $\langle h, \bar{\varepsilon}\rangle$ *for the scalar optimization problem*

$$\text{Min}\{\langle h, \Psi(x, \tilde{s}^1_0, \tilde{s}^2_0)\rangle | x \in K\},$$

where the feasible set is $K = S \cap Q \cap M \cap \{x \in X | f(x) - f(x_0) \in -\mathbb{R}^p_+\}$. *Then* $x_0 \in \text{Min}_{\langle h, \bar{\varepsilon} - \tilde{s}^1_0(g(x_0))\rangle, h}(f, S \cap Q \cap M)$.

Proof. Suppose that the conclusion is false. Then there exists a point $x \in S \cap Q \cap M$ such that $f(x) - f(x_0) \in -\mathbb{R}^p_+$ and $\langle h, f(x_0)\rangle > \langle h, f(x)\rangle + \langle h, \bar{\varepsilon} - \tilde{s}^1_0(g(x_0))\rangle$. Hence $x \in K$, and we have $\tilde{s}^1_0(g(x)) \in -\mathbb{R}^p_+$ and $A(x) - b = 0$ since $\tilde{s}^1_0 \in L^+_m$ and $x \in S \cap Q$. Moreover, $A(x_0) - b = 0$ because $x_0 \in Q$. Then

$$\langle h, f(x_0) + \tilde{s}^2_0(A(x_0) - b)\rangle > \langle h, f(x)\rangle + \langle h, \bar{\varepsilon} - \tilde{s}^1_0(g(x_0))\rangle \geqslant \langle h, f(x) + \tilde{s}^1_0(g(x)) + \tilde{s}^2_0(A(x) - b)\rangle + \langle h, \bar{\varepsilon} - \tilde{s}^1_0(g(x_0))\rangle. \tag{42}$$

Since $x, x_0 \in M$ and $\tilde{s}_0^1 \in L_m^+$, (42) shows that

$$\langle h, \Psi(x_0, \tilde{s}_0^1, \tilde{s}_0^2)\rangle - \langle h, \bar{\varepsilon}\rangle > \langle h, \Psi(x, \tilde{s}_0^1, \tilde{s}_0^2)\rangle.$$

This is a contradiction because $x_0 \in \mathrm{Min}_{\langle h,\bar{\varepsilon}\rangle}(\langle h, \Psi(\cdot, \tilde{s}_0^1, \tilde{s}_0^2)\rangle, K)$. □

The proof of Proposition 4.8 shows that the same result is true for the Lagrangian function Φ_{d_1,d_2} if $\tilde{s}_0^1 \in L_{m,d_1}^+$.

5. Conclusions

In this work we have studied Helbig's approximate solutions in convex Pareto problems. Our first objective has been to obtain multiplier rules for these solutions. In order to attain this objective we have related the multiobjective problem with a scalar minimax program. This procedure has proved to be very useful for our propose.

Next we have deduced the corresponding approximate saddle-point theorems using two different vector-valued Lagrangian functions. Here our development closely follows the line of classical Lagrangian saddle-point results starting from the Kuhn–Tucker conditions previously obtained. Our Corollary 4.7 gives an answer to a problem formulated by Vályi (1987) in Remark 3.3.

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